Tutorial Notes 7

1. f is in $C^1[a,b]$ and positive. Consider the curve $\gamma(t) = (t, f(t))$ and the force $F = (y, 0)$. Prove that the work of F along γ is equal to the area of the region below the function f.

   Solutions:

   The work is
   $$\int_a^b (f(t), 0) \cdot (1, f'(t))\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t,$$
   which is the area of the region below f.

2. A particle moves from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is
   $$k[(b^2 + f(b)^2)^{1/2} - (a^2 + f(a)^2)^{1/2}].$$

   Solutions:

   Suppose that the arc-length parametrization of the path is $\gamma(s)$, $0 \le s \le L$. Then the force is $k\gamma(s)$ and the work is
   $$\int_0^L k\gamma(s) \cdot \dot{\gamma}(s)\,\mathrm{d}s = k\left.\frac{|\gamma(s)|^2}{2}\right|_0^L = k[(b^2 + f(b)^2)^{1/2} - (a^2 + f(a)^2)^{1/2}].$$

3. Find the potential functions for the vector field
   $$\left(\frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}, \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right).$$

   Solutions:

   We only need to solve the following equations:
   $$\frac{\partial \varphi}{\partial x} = \frac{y}{1+x^2y^2}; \tag{1a}$$
   $$\frac{\partial \varphi}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}; \tag{1b}$$
   $$\frac{\partial \varphi}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}. \tag{1c}$$

   By (1a),
   $$\varphi(x,y,z) = \arctan(xy) + f(y,z).$$

Substituting φ into (1b), it follows that

$$f(y,z) = \arcsin(yz) + g(z).$$

Substituting φ into (1c), we have

$$g(z) = \log z + C.$$

Therefore,

$$\varphi = \arctan(xy) + \arcsin(yz) + \log z + C.$$

4. Show that the differential form in the following integral is exact and evaluate the integral:

$$\int_{(1,0,0)}^{(0,1,1)} (\sin y \cos x \,\mathrm{d}x + \cos y \sin x \,\mathrm{d}y + \mathrm{d}z).$$

Solutions:

First we find the potential functions. It suffices to solve the following equations:

$$\frac{\partial \varphi}{\partial x} = \sin y \cos x; \tag{2a}$$

$$\frac{\partial \varphi}{\partial y} = \cos y \sin x; \tag{2b}$$

$$\frac{\partial \varphi}{\partial z} = 1. \tag{2c}$$

By (2a),

$$\varphi(x,y,z) = \sin y \sin x + f(y,z).$$

Substituting φ into (2b), it follows that

$$f(y,z) = g(z).$$

Substituting φ into (2c), we have

$$g(z) = z + C.$$

Hence the potential functions are

$$\sin y \sin x + z + C$$

and the differential form in the integral is exact. Moreover, the integral is

$$(\sin y \sin x + z + C)\big|_{(0,1,1)}^{(1,0,0)} = -1.$$

5. Find the potential functions for the vector field in the following integral and evaluate the integral:

$$\int_{(1,2,1)}^{(2,1,1)} \left[(2x\log y - yz)\,\mathrm{d}x + \left(\frac{x^2}{y} - xz\right)\mathrm{d}y - xy\,\mathrm{d}z\right].$$

Solutions:

First we find the potential functions. It suffices to solve the following equations:

$$\frac{\partial \varphi}{\partial x} = 2x \log y - yz; \tag{3a}$$

$$\frac{\partial \varphi}{\partial y} = \frac{x^2}{y} - xz; \tag{3b}$$

$$\frac{\partial \varphi}{\partial z} = -xy. \tag{3c}$$

By (3a),

$$\varphi(x, y, z) = x^2 \log y - xyz + f(y, z).$$

Substituting φ into (3b), it follows that

$$f(y, z) = g(z).$$

Substituting φ into (3c), we have

$$g(z) = C.$$

Hence the potential functions are

$$x^2 \log y - xyz + C.$$

Moreover, the integral is

$$(x^2 \log y - xyz + C)\Big|_{(1,2,1)}^{(2,1,1)} = -\log 2.$$